International Journal of Trend in Scientific Research and Development (IJTSRD)
Volume 8 Issue 1, January-February 2024 Available Online: www.ijtsrd.com e-ISSN: 2456 – 6470

H^1-L^1 Boundedness of Rough Toroidal Pseudo-Differential Operator

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ABSTRACT

In this paper, we study rough toroidal pseudo-differential operators. We prove that they are bounded from H^1 to L^1 on torus for symbols in a critical rough toroidal class.

KEYWORDS: *Rough toroidal pseudo-differential operator; Rough toroidal symbol; H^1 (T^n); Torus; Atomic decomposition*

Mathematics Subject Classification
Primary: 42B05,
Secondary: 35S05.

How to cite this paper: Ramla Benhamoud "H1-L1 Boundedness of Rough Toroidal Pseudo-Differential Operator" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456-6470, Volume-8 | Issue-1, February 2024, pp.263-270, URL: www.ijtsrd.com/papers/ijtsrd62379.pdf

IJTSRD62379

INTRODUCTION

The theory of pseudo-differential operators was started by Kohn and Nirenberg and Hörmander. These operators have wide use in the study of partial differential equations and harmonic analysis. They are defined on $\mathbb{R}^n$ as follows

$$T_a f(x) = \int_{\mathbb{R}^n} e^{ix\xi}\, a(x,\xi)\hat{f}(\xi)d\xi,$$

where $\hat{f}$ is the Fourier transform of f and $a(x,\xi)$ is called the symbol of the operator T_a. According to the behavior of symbols and their derivatives, there are different symbol classes.

Let $\mathbb{N}$ be the set $\{1,2,3,\dots\}$ and $\mathbb{N}_0$ the set $\{0,1,2,\dots\}$. We recall the classical symbol class introduced in which is known by Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)(m\in\mathbb{R}$ and $0\le\rho,\delta\le 1)$. It is the set of functions a that are smooth in (x,ξ) and satisfy

$$|\nabla_\xi^N \nabla_x^M a(x,\xi)| \le C(1+|\xi|)^{m-\rho N+\delta M}$$

for any $N, M \in \mathbb{N}_0$.

The concept of periodic pseudo-differential operators was introduced by Agranovich. He presented a global quantization on the unit circle $\mathbb{S}^1$ that could be extended to the torus $\mathbb{T}^n$. These operators can be associated with the symbols $a\colon \mathbb{T}^n\times\mathbb{R}^n \mapsto \mathbb{C}$ or $a\colon \mathbb{T}^n\times\mathbb{R}^n \mapsto \mathbb{C}$ and denoted by $a(x,D)$. The equivalence between them has been proven in .

Let the torus $\mathbb{T}^n=(\mathbb{R}/2\pi\mathbb{Z})^n$ and $\mathcal{S}(\mathbb{Z}^n)$ be the restriction of schwartz space $\mathcal{S}(\mathbb{R}^n)$ on $\mathbb{Z}^n$. We define the toroidal Fourier transform $\mathcal{F}_{\mathbb{T}^n}\colon C^\infty(\mathbb{T}^n)\to\mathcal{S}(\mathbb{Z}^n)$ by

$$(\mathcal{F}_{\mathbb{T}^n} f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n}\int_{\mathbb{T}^n} e^{-ix\xi} f(x)dx.$$

Then, $\mathcal{F}_{\mathbb{T}^n}$ is a bijection and its inverse $\mathcal{F}^{-1}_{\mathbb{T}^n}\colon \mathcal{S}(\mathbb{Z}^n)\to C^\infty(\mathbb{T}^n)$ is given by

$$f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{ix\xi}\,(\mathcal{F}_{\mathbb{T}^n} f)(\xi) = \sum_{\xi\in\mathbb{Z}^n} e^{ix\xi}\,\hat{f}(\xi).$$

In , Cardona et al. defined the periodic Hörmander class $S^m_{\rho,\delta}(\mathbb{T}^n\times\mathbb{R}^n)$ as a class of Hörmander symbols $S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ which are 2π −periodic in x, i.e., the set of smooth functions in $(x,\xi)\in\mathbb{T}^n\times\mathbb{R}^n$ and satisfy

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}.$$

The corresponding periodic pseudo-differential operator is given by

$$a(x,D)f(x) = (2\pi)^{-n}\int_{\mathbb{T}^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi}\, a(x,\xi)f(y)d\xi dy.$$

Let Δ_{ξ_j}, $\xi \in \mathbb{Z}^n$, $j = 1,\ldots,n$ be the forward partial difference operators such that for each function $f: \mathbb{Z}^n \to \mathbb{C}$

$$\Delta_{\xi_j} f(\xi) = f(\xi + \delta_j) - f(\xi),$$

and for $\alpha \in \mathbb{N}_0^n$

$$\Delta_\xi^\alpha = \Delta_{\xi_1}^{\alpha_1}\ldots\Delta_{\xi_n}^{\alpha_n}.$$

The toroidal class $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$, $0 \le \rho, \delta \le 1$ is the set of functions $a(x,\xi)$ that are smooth in x for all $\xi \in \mathbb{Z}^n$ and satisfy

$$|\Delta_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$. The toroidal pseudo-differential operators are defined by

$$a(x,D)f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{ix\xi}\, a(x,\xi)\hat{f}(\xi), \quad f \in C^\infty(\mathbb{T}^n).$$

Turunen and Vainikko studied products and asymptotic expansions for adjoints of periodic pseudo-differential operators. More results about the calculations of these operators have been presented by Ruzhansky and Turunen in and .

The boundedness of pseudo-differential operators for symbols in Hörmander class $S^m_{\rho,\delta}$ is an interesting topic that has been extensively studied in the theory of these operators. We refer to for results on $\mathbb{R}^n$. Many studies have been conducted to explore the boundedness of periodic pseudo-differential operators on torus $\mathbb{T}^n$ and Lie groups. For instance, when symbols with finite regularity in $x \in \mathbb{T}^n$, Ruzhansky and Turunen proved the L^2 boundedness for $|\partial_x^\beta a(x,\xi)| \le C$ for all $(x,\xi) \in \mathbb{T}^n \times \mathbb{Z}^n$ and $|\beta| \le n + 1$ and without any regularity require for the variable ξ which is better than results on $\mathbb{R}^n$ since the torus $\mathbb{T}^n$ is compact. In , the L^2 boundedness has been shown for $a \in S^0_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)(0 \le \delta < \rho \le 1)$. The same result has been proved by Fischer for the compact Lie group G. Delagdo studied L^p boundedness, he showed that for $a \in S^{n/2(\rho-1)}_{\rho,0}(\mathbb{T}^n \times \mathbb{Z}^n)$, $0 < \rho < 1$, the periodic pseudo-differential operator is bounded from L^∞ into BMO, and by interpolation he concluded $L^p(2 \le p < \infty)$ boundedness. This result was extended to compact Lie group G in. Delagdo and Ruzhansky extended also L^p boundedness for $1 < p < 2$ by using duality. They proved also for symbols with finite regularity that the operator $a(x,D)$ is bounded on $L^p(\mathbb{T}^n)$ $(1 < p < \infty)$ for $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$, $m = -\left[\frac{n}{2}\right](1 - \rho)|1/p - 1/2| - \delta\left(\left[\frac{n}{p}\right] + 1\right)$ and $|\beta| \le \left[\frac{n}{p}\right] + 1$, where $0 \le \delta < \rho < 1$. For more results concerning $L^p(\mathbb{T}^n)(1 < p < \infty)$ boundedness, one can see .

The weak type (1,1) was studied in , Cardona proved that for $a \in S^0_{1,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$, $0 \le \delta < 1$, the pseudo-differential operator is weak type (1,1) on torus $\mathbb{T}^n$. For the multi-linear pseudo-differential operators on $\mathbb{T}^n \times \mathbb{Z}^n$, we refer to and references therein. Furthermore, in , Cardona et al. showed that the boundedness of Fourier integral operators in $L^p(\mathbb{R}^n)(1 < p < \infty)$ implies the boundedness of periodic Fourier integral operators in $L^p(\mathbb{T}^n)(1 < p < \infty)$, i.e., for pseudo-differential operators is also true.

In , Kenig and Staubach defined a rough Hörmander symbol class $L^\infty S^m_\rho$ which symbols behave in the spatial variable like an L^∞ function. It is known that $S^m_{\rho,1} \subset L^\infty S^m_\rho$. Recently, Guo and Zhu studied endpoint estimates for the symbol class $a \in S^{n(\rho-1)}_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and rough class $L^\infty S^{n(\rho-1)}_\rho$. We recall their Theorem 1.2.

Theorem A 1. *Let $a \in L^\infty S^{n(\rho-1)}_\rho(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \le \rho < 1$. Then the operator T_a is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

For $\rho = 1$ they constructed a symbol $a \in S^0_{1,1}$ such that T_a is unbounded from H^1 to L^1. Similarly, we state the definition of rough toroidal symbol class $L^\infty S^m_\rho(\mathbb{T}^n \times \mathbb{Z}^n)$.

Definition 1. *The class $L^\infty S^m_\rho(\mathbb{T}^n \times \mathbb{Z}^n)$, $0 \le \rho \le 1$ is the set of functions which are measurable in x and satisfy $\| \Delta_\xi^\alpha a(\cdot,\xi) \|_{L^\infty(\mathbb{T}^n)} \lesssim \langle\xi\rangle^{m-\rho|\alpha|}$.*

In this note, we consider the rough toroidal class $L^\infty S^m_\rho(\mathbb{T}^n \times \mathbb{Z}^n)$ and we obtain a periodic version of Theorem A. Our main result is the following:

Theorem 1. *If $a \in L^\infty S^{n(\rho-1)}_\rho(\mathbb{T}^n \times \mathbb{Z}^n)$, $0 \le \rho < 1$, then the toroidal pseudo-differential operator $a(x,D)$ is bounded from $H^1(\mathbb{T}^n)$ to $L^1(\mathbb{T}^n)$.*

Remark 1. *Theorem 1 extends the symbol class where the pseudo-differential operators are $H^1 - L^1$ bounded on torus to $S^{n(\rho-1)}_{\rho,\delta}, 0 \le \rho < 1$, and $0 \le \delta \le 1$. Our proof is based on the proprieties of the difference operator, i.e., we follow the method used in by using the difference operator instead of the derivative.*

This note is organized as follows. In Section 2, we recall some basic notions and lemmas about the periodic analysis of pseudo-differential operators and the atomic characterization of Hardy spaces on a torus $\mathbb{T}^n$. In Section 3, we prove Theorem 1.

Preliminary notions and lemmas

The following lemma is a periodic version of Proposition 2.3 in.

Lemma 1. *Assume* $0 \leq \rho \leq 1$, $a \in L^{\infty} S_{\rho}^{n(\rho-1)}(\mathbb{T}^n \times \mathbb{Z}^n)$ *and* $1 < p < \infty$. *Then the operator* $a(x, D)$ *is bounded from* $L^p(\mathbb{T}^n)$ *to* $L^p(\mathbb{T}^n)$.

Proof. Since from , the $L^p(1 < p < \infty)$ boundedness of pseudo-differential operators on $\mathbb{R}^n$ implies the boundedness of periodic operators on $\mathbb{T}^n$. Then we get directly Lemma 1 from. □

For two multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, $\beta \leq \alpha$ means that $\beta_k \leq \alpha_k$ for all $1 \leq k \leq n$ and we denote

$$\binom{\alpha}{\beta} = \prod_{k=1}^{n} \binom{\alpha_k}{\beta_k} = \prod_{k=1}^{n} \frac{\alpha_k!}{(\alpha_k - \beta_k)!\,\beta_k!}.$$

We recall some known proprieties for the difference operator Δ_ξ.

Lemma 2. *() Let* $f, g: \mathbb{Z}^n \to \mathbb{C}$. *There holds* $\Delta_\xi^\alpha (fg)(\xi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \Delta_\xi^\beta f \Delta_\xi^{\alpha-\beta} g(\xi + \beta)$.

Lemma 3. *Let* $f, g: \mathbb{Z}^n \to \mathbb{C}$. *Then* $\sum_{\xi \in \mathbb{Z}^n} f(\xi) \left[\Delta_\xi^\gamma g\right](\xi) = (-1)^\gamma \sum_{\xi \in \mathbb{Z}^n} \left[\bar{\Delta}_\xi^\gamma f\right](\xi) g(\xi)$, *where* $\bar{\Delta}_{\xi_j} f(\xi) = f(\xi) - f(\xi - \delta_j)$.

Atomic decomposition for $H^p(\mathbb{T}^n)$:

As to the Euclidean case $\mathbb{R}^n$, Hardy space on a torus $\mathbb{T}^n (n \geq 1)$ admits atomic decomposition. For more details, we refer to and , where Folland and Stein showed atomic decomposition for Hardy spaces on homogeneous groups and in Qing et al. for Hardy spaces on graded Lie groups. And for $n = 1$, we can see .

Definition 2. *Let* $0 < p \leq 1 \leq q \leq \infty$, *$p\neq q\textcolor{red}{,}$ and* s *is an integer at least* $[n(1/p - 1)]$. *We say a function* b *on* $\mathbb{T}^n$ *is a* (p, q, s)*-atom if it is a compactly supported* L^q *function such that*

1. *there is a ball* B *such that* $\operatorname{supp} b \subset \bar{B}$ *and $\|b\|_{L^{q}}\leq\lvert B\lvert^{1/q-1/p}\textcolor{red}{.}$*
2. $\int_{\mathbb{T}^n} b(x) x^\alpha dx = 0$, *where* $0 \leq |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \leq s$ *and* $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$.

The atomic decomposition of $H^p(\mathbb{T}^n)$ is given as follows.

Proposition 1. *Let* (p, q, s) *be as in Definition 2. Then for all any* (p, q, s) *atom* a, *one has* $\| a \|_{H^p(\mathbb{T}^n)} \lesssim 1$. *Conversely, for any* $f \in H^p(\mathbb{T}^n)$, *there exists a sequence* $\{\lambda_j\}_{j=1}^\infty$ *such that* $f = \sum_{j=1}^\infty \lambda_j a_j$ *with convergence in* $\mathcal{S}'(\mathbb{T}^n)$ *and* $\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{1/p} \lesssim \| f \|_{H^p(\mathbb{T}^n)}$.

Before giving the proof of our main result, we present the following inequality which will be used in our proof.

Lemma 4. *For any* L^2*-atom* b *for* $H^1(\mathbb{T}^n)$ *and* $n \geq 1$, *we have,* $\sum_{\xi \in \mathbb{Z}^n - \{0\}} |\hat{b}(\xi)| |\xi|^{-n} \leq C$.

Proof. From Definition 2, we have $\operatorname{supp} b \subset B(x_0, r)$, $\int_{B(x_0,r)} b(x) dx = 0$ and $\| b \|_{L^2} \leq r^{-n/2}$.

Case 1. If $r < 1$, by using Hölder inequality and the inequality in $|e^{-ix\xi} - 1| \lesssim |x|^\theta |\xi|^\theta$ which holds for all $\theta \in (0,1)$, we can get

$$\begin{aligned}
\sum_{\xi \in \mathbb{Z}^n - \{0\}} |\hat{b}(\xi)| |\xi|^{-n} &= \sum_{|\xi| \leq [r^{-1}]} |\hat{b}(\xi)| |\xi|^{-n} + \sum_{|\xi| > [r^{-1}]} |\hat{b}(\xi)| |\xi|^{-n} \\
&\lesssim \sum_{|\xi| \leq [r^{-1}]} |\xi|^{-n} \int_{B(x_0,r)} |e^{-ix\xi} - 1| |b(x)| dx + \sum_{|\xi| > [r^{-1}]} |\hat{b}(\xi)| |\xi|^{-n} \\
&\lesssim \sum_{|\xi| \leq [r^{-1}]} |\xi|^{\theta - n} \int_{B(x_0,r)} |x|^\theta |b(x)| dx + \left(\sum_{|\xi| > [r^{-1}]} |\hat{b}(\xi)|^2\right)^{1/2} \left(\sum_{|\xi| > [r^{-1}]} |\xi|^{-2n}\right)^{1/2} \\
&\lesssim \sum_{|\xi| \leq [r^{-1}]} |\xi|^{\theta - n} \| b \|_{L^2} \left(\int_{B(x_0,r)} |x|^{2\theta} dx\right)^{1/2} + \| b \|_{L^2} \left(\sum_{|\xi| > [r^{-1}]} |\xi|^{-2n}\right)^{1/2} \\
&\lesssim \sum_{|\xi| \leq [r^{-1}]} |\xi|^{\theta - n} r^{-n/2} r^{3n\theta/2} + r^{-n/2} \left(\sum_{|\xi| > [r^{-1}]} |\xi|^{-2n}\right)^{1/2} \\
&\lesssim \sum_{|\xi| \leq [r^{-1}]} |\xi|^{\theta - n} r^{-n/2} r^{3n\theta/2} + r^{-n/2} r^{n/2} \left(\sum_{|\xi| > [r^{-1}]} |\xi|^{-n}\right)^{1/2} \\
&\lesssim 1
\end{aligned}$$

we can take $\theta = 1/3$.

Case 2. If $r \geq 1$ we can get

$$\sum_{\xi \in \mathbb{Z}^n - \{0\}} |\hat{b}(\xi)| |\xi|^{-n} \leq \left(\sum_{\xi \in \mathbb{Z}^n - \{0\}} |\hat{b}(\xi)|^2\right)^{1/2} \left(\sum_{\xi \in \mathbb{Z}^n - \{0\}} |\xi|^{-2n}\right)^{1/2} \lesssim 1.$$

Proof of our main result

To start proving Theorem 1, we use atomic decomposition for $H^1(\mathbb{T}^n)$. It is enough to show $\| a(x,D)b \|_{L^1(\mathbb{T}^n)} \lesssim 1$ for a L^2-atom b. We prove the case $\rho = 0$ latter, since the kernel is not rapidly decreasing. First, we consider $0 < \rho < 1$.

Case of $\boldsymbol{0 < \rho < 1}$

We denote the kernel of $a(x,D)$ by

$$k(x,y) = \sum_{\xi \in \mathbb{Z}^n} e^{i(x-y)\xi} a(x,\xi).$$

From summation by parts in Lemma 3, one can easily show the equality

$$\left(e^{-i(x-y)} - 1\right)^{\gamma} e^{i(x-y)\xi} = (-1)^{|\gamma|} \bar{\Delta}_{\xi}^{\gamma} e^{i(x-y)\xi},$$

and since we are working on the torus $\mathbb{T}^n$, it is easy to check that the following inequality in

$$|x-y|^N \leq c \sum_{|\gamma|=N} |\left(e^{-i(x-y)} - 1\right)^{\gamma}|,$$

is true for all $x, y \in \mathbb{T}^n$. From [3.1] and [3.2], it yields

$$\begin{aligned}
\left(e^{-i(x-y)} - 1\right)^{\gamma} k(x,y) &= \sum_{\xi \in \mathbb{Z}^n} \left(e^{-i(x-y)} - 1\right)^{\gamma} e^{i\xi(x-y)} a(x,\xi) \\
&= \sum_{\xi \in \mathbb{Z}^n} (-1)^{\gamma} \left[\bar{\Delta}_{\xi}^{\gamma} e^{i\xi(x-y)}\right] a(x,\xi) \\
&= \sum_{\xi \in \mathbb{Z}^n} e^{i\xi(x-y)} \Delta_{\xi}^{\gamma}[a(x,\xi)].
\end{aligned}$$

On the other hand, let $B_j = \{\xi \in \mathbb{Z}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and $|\xi|_\infty = \max_i |\xi_i|$ where ξ_i represents the i-th component of the vector $\xi \in \mathbb{Z}^n$. It is clear that $\mathbb{Z}^n = \bigcup_{j \in \mathbb{N}_0} B_j$. If we choose $\mu = 2^{j+1}$ and $0 < \epsilon < 1$, we can write

$$\begin{aligned}
\sum_{\xi \in B_j} \langle \xi \rangle^{\epsilon-n} &\lesssim \sum_{l \in \mu^{-1}\mathbb{Z}^n \setminus \{0\}, |l|_\infty \leq 1} |\mu l|^{\epsilon-n} \\
&\lesssim \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}, |\xi|_\infty \leq \mu} |\xi|^{\epsilon-n} \\
&\lesssim \sum_{N=1}^{[\mu]+1} \sum_{\xi \in Y_N} |\xi|^{\epsilon-n},
\end{aligned}$$

such that

$$\begin{aligned}
\mathbb{Z}^n \setminus \{0\} &= \cup_{N \in \mathbb{N}} \{\xi \in \mathbb{Z}^n : |\xi|_\infty = N\} \\
&= \cup_{N \in \mathbb{N}} Y_N
\end{aligned}$$

and $card\, Y_N \leq C N^{n-1}$. Then, we can obtain

$$\begin{aligned}
\sum_{\xi \in B_j} \langle \xi \rangle^{\epsilon-n} &\lesssim \sum_{N=1}^{[\mu]+1} N^{n-1} N^{\epsilon-n} \\
&\lesssim \int_0^{[\mu]} x^{\epsilon-1}\, dx \\
&\lesssim 2^{j\epsilon}.
\end{aligned}$$

So, by some simple computations, we can estimate the kernel $k(x,y)$ as follows

$$\begin{aligned} |k(x,y)| &\lesssim |e^{i(x-y)}-1|^{-N}\sum_{\xi\in\mathbb{Z}^n}|\Delta_\xi^N a(x,\xi)| \\ &\lesssim |x-y|^{-N}\sum_{\xi\in\mathbb{Z}^n}\langle\xi\rangle^{n(\rho-1)-N\rho} \\ &\lesssim |x-y|^{-N}\sum_{j\in\mathbb{N}_0}\sum_{\xi\in B_j}\langle\xi\rangle^{n(\rho-1)-N\rho} \\ &\lesssim |x-y|^{-N}\sum_{j\in\mathbb{N}_0}2^{jn(\rho-1)-jN\rho+jn-j\epsilon}\sum_{\xi\in B_j}\langle\xi\rangle^{\epsilon-n} \\ &\lesssim |x-y|^{-N}\sum_{j\in\mathbb{N}_0}2^{jn\rho-jN\rho} \\ &\lesssim |x-y|^{-N} \end{aligned}$$

the last inequality is true for $N>n$. Then, if $|x|>2r$, it yields

$$|a(x,D)b(x)| = \left|\int_{|y|\le r} k(x,y)b(y)dy\right| \lesssim \int_{|y|\le r}|x-y|^{-N}|b(y)|dy \lesssim |x|^{-N}.$$

Part 1

Assume $r\ge 1$, then

$$\begin{aligned} \| a(x,D)b \|_{L^1(\mathbb{T}^n)} &= \int_{\mathbb{T}^n\setminus B(0,2r)}|a(x,D)b(x)|dx + \int_{B(0,2r)}|a(x,D)b(x)|dx \\ &\lesssim \int_{\mathbb{T}^n\setminus B(0,2r)}|x|^{-N}dx + r^{n/2}\| a(x,D)b \|_2 \\ &\lesssim 1 + r^{n/2}\| b \|_2 \lesssim 1. \end{aligned}$$

Part 2

For $r<1$, we use discrete dyadic decomposition $(\varphi_j)_{j\in\mathbb{N}_0}\subset\mathcal{S}(\mathbb{Z}^n)$ as in .

- $supp\ \varphi_0 \subset \{\xi\in\mathbb{Z}^n: |\xi|\le 2\}$ and $\varphi_0(\xi)=1$ when $|\xi|\le 1$.
- $supp\ \varphi_j \subset \{\xi\in\mathbb{Z}^n: 2^{j-1}<|\xi|\le 2^{j+1}\}$ for $j\ge 1$.
- For any $\xi\in\mathbb{Z}^n$, one have $0\le\varphi_j(\xi)\le 1$ $(j\in\mathbb{N}_0)$ and $\sum_{j\in\mathbb{N}_0}\varphi_j(\xi)=1$.
- For any $\alpha\in\mathbb{N}_0^n$, there exists a constant $C_\alpha>0$ such that

$$\left|\Delta_\xi^\alpha\varphi_j(\xi)\right| \le C_\alpha\langle\xi\rangle^{-|\alpha|}$$

for any $j\in\mathbb{N}, \xi\in\mathbb{Z}^n$.

Then, we can write

$$\begin{aligned} a(x,D)b(x) &= \sum_{j\in\mathbb{N}_0}\sum_{\xi\in B_j}\int_{\mathbb{T}^n}e^{i(x-y)\xi}a(x,\xi)\varphi_j(\xi)b(y)dy \\ &= \sum_{j\in\mathbb{N}_0}a_j(x,D)b(x), \end{aligned}$$

where $B_j=\{\xi\in\mathbb{Z}^n: 2^{j-1}\le|\xi|\le 2^{j+1}\}$ and $B_0=\{\xi\in\mathbb{Z}^n: |\xi|\le 2\}$.

Using the L^2 boundedness result in Lemma 1, we can get

$$\int_{|x|\le 3r}\Big|\sum_{j\in\mathbb{N}_0}a_j(x,D)b(x)\Big|dx \lesssim 1.$$

Now we shall prove

$$\int_{|x|>3r}|a_j(x,D)b(x)|dx \lesssim 1.$$

Assume $2^j \geq r^{-1/\rho}$, one can get

$$\begin{aligned}
\sum_{2^j \geq r^{-1/\rho}} \int_{|x|>3r} |a_j(x,D)b(x)|dx &\lesssim \sum_{2^j \geq r^{-1/\rho}} \int_{|x|>3r} \int_{|y|\leq r} |e^{i(x-y)}-1|^{-N} 2^{j\rho(n-N)} |b(y)|dydx \\
&\lesssim \sum_{2^j \geq r^{-1/\rho}} \int_{|x|>3r} 2^{j\rho(n-N)} \int_{|y|\leq r} |x-y|^{-N}|b(y)|dydx \\
&\lesssim \sum_{2^j \geq r^{-1/\rho}} \int_{|x|>3r} 2^{j\rho(n-N)} |x|^{-N} dx \\
&\lesssim 1.
\end{aligned}$$

For $2^j < r^{-1/\rho}$, we have

$$\sum_{2^j < r^{-1/\rho}} \int_{|x|>3r} |a_j(x,D)b(x)|dx \lesssim \sum_{2^j < r^{-1/\rho}} \int_{|x|>3r} \sum_{\xi \in B_j} |x|^{-N} |\Delta_\xi^N [a(x,\xi)\varphi_j(\xi)\hat{b}(\xi)]|.$$

From discrete Leibniz formula Lemma 2, and for $\alpha \neq N$, we can obtain

$$\begin{aligned}
|\Delta_\xi^{N-\alpha}\hat{b}(\xi+\alpha)| &= |\Delta_\xi^{N-\alpha}\left(\int_{\mathbb{T}^n} e^{-iy(\xi+\alpha)}\, b(y)dy\right)| \\
&\leq \int_{\mathbb{T}^n} |y|^{N-\alpha}|b(y)|dy \\
&\leq \int_{|y|\leq r} r^{N-\alpha}\,|b(y)|dy \\
&\lesssim r^{N-\alpha}.
\end{aligned}$$

Then, we can get the following estimate

$$\begin{aligned}
&|\Delta_\xi^N [a(x,\xi)\varphi_j(\xi)\hat{b}(\xi)]| \\
&\leq \sum_{0\leq\alpha\leq N} |\Delta_\xi^\alpha [a(x,\xi)\varphi_j(\xi)]||\Delta_\xi^{N-\alpha}\hat{b}(\xi+\alpha)| \\
&= |\Delta_\xi^N [a(x,\xi)\varphi_j(\xi)]||\hat{b}(\xi+N) + \sum_{0\leq\alpha<N} 2^{jn(\rho-1)-\rho\alpha}\, r^{N-\alpha} \\
&\lesssim 2^{jn(\rho-1)-j\rho N}|\hat{b}(\xi+N)| + \sum_{0\leq\alpha<N} 2^{jn(\rho-1)}\, r^N \left(2^{-j\rho}r^{-1}\right)^\alpha \\
&\lesssim 2^{jn(\rho-1)-j\rho N}|\hat{b}(\xi+N)| + 2^{jn(\rho-1)}2^{-j\rho N}2^{j\rho}r \\
&\lesssim 2^{jn(\rho-1)-j\rho N}\left[|\hat{b}(\xi+N)| + 2^{j\rho}r\right].
\end{aligned}$$

So, from Lemma 4 we obtain

$$\begin{aligned}
&\sum_{2^j < r^{-1/\rho}} \int_{|x|>3r} |a_j(x,D)b(x)|dx \\
&\leq \sum_{2^j < r^{-1/\rho}} \int_{|x|>3r} \sum_{\xi\in B_j} |x|^{-N} 2^{jn(\rho-1)-j\rho N}\left[\hat{b}(\xi+N) + 2^{j\rho}r\right]dx \\
&\lesssim \sum_{2^j < r^{-1/\rho}} \sum_{\xi\in B_j} 2^{jn(\rho-1)-j\rho N}\left[|\hat{b}(\xi+N)| + 2^{j\rho}r\right] \\
&\lesssim \sum_{2^j < r^{-1/\rho}} \sum_{\xi\in B_j} 2^{jn(\rho-1)-j\rho N}\,|\hat{b}(\xi+N)| + \sum_{2^j < r^{-1/\rho}} \sum_{\xi\in B_j} 2^{jn(\rho-1)-j\rho N}\, 2^{j\rho}r \\
&\lesssim \sum_{2^j < r^{-1/\rho}} \sum_{\xi\in B_j} 2^{jn(\rho-1)-j\rho N}\,|\hat{b}(\xi+N)| + \sum_{2^j < r^{-1/\rho}} 2^{j\rho(n-N)}\, 2^{j\rho}r \\
&\lesssim \sum_{2^j < r^{-1/\rho}} \sum_{\xi\in B_j} 2^{j\rho n-j\rho N}\,|\hat{b}(\xi+N)||\xi|^{-n} + \sum_{2^j < r^{-1/\rho}} 2^{j\rho(n-N)}\, 2^{j\rho}r \\
&\lesssim 1.
\end{aligned}$$

Case of $\rho = 0$

In this part, we shall show that the operator $a(x,D)$ is bounded from H^1 to L^1 for the symbol $a \in L^\infty S_0^{-n}(\mathbb{T}^n \times \mathbb{Z}^n)$. As in the case $0 < \rho < 1$ we first get the estimate for $r \geq 1$ then $r < 1$.
Part 1. Suppose $r \geq 1$, if $x < 2r$, from Lemma 1 we can get

$$\| a(x,D)b \|_{L^1(B(x_0,2r))} \lesssim r^{n/2} \| a(x,D)b \|_{L^2(\mathbb{T}^n)} \lesssim r^{n/2} \| b \|_{L^2(\mathbb{T}^n)} \lesssim 1.$$

For $\{x \in \mathbb{T}^n : |x| \geq 2r\}$, we use summation by parts in Lemma 2

$$\begin{aligned} |a(x,D)b(x)| &= |x|^{-n} |\sum_{\xi\in\mathbb{Z}^n} e^{ix\xi} \Delta_\xi^n \left(a(x,\xi)\hat{b}(\xi)\right)| \\ &= |x|^{-n} \sum_{l=0}^{n} |\sum_{\xi\in\mathbb{Z}^n} e^{ix\xi} \Delta_\xi^{n-l} a(x,\xi)\widehat{y^l b}(\xi)| \\ &= |x|^{-n} \sum_{l=0}^{n} | a(x,D)[y^l b](x)| \end{aligned}$$

where $a_l(x,D)$ is a periodic pseudo-differential operator with symbol $\Delta_\xi^{n-l} a(x,\xi)$. It is obvious that $\Delta_\xi^{n-l} a(x,\xi) \in L^\infty S_0^{-n}(\mathbb{T}^n \times \mathbb{Z}^n)$. Since from Lemma 1, the operator $a_l(x,D)$ is bounded on $L^2(\mathbb{T}^n)$, one can obtain

$$\begin{aligned} \int_{\mathbb{T}^n \backslash B(x_0,2r)} | a(x,D)b(x)| dx &\lesssim \sum_{l=0}^{n} \int_{\mathbb{T}^n \backslash B(x_0,2r)} | x|^{-n} |a_l(x,D)(y^l b)(x)| dx \\ &\lesssim \sum_{l=0}^{n} r^{-n/2} \| y^l b \|_{L^2(\mathbb{T}^n)} \\ &\lesssim \sum_{l=0}^{n} r^{-n+l} \lesssim 1. \end{aligned}$$

Part 2. For $r < 1$, applying Lemma 4, it yields

$$|a(x,D)b(x)| = \sum_{\xi\in\mathbb{Z}^n} e^{ix\xi} a(x,\xi)\hat{b}(\xi) d\xi| \leq \sum_{\xi\in\mathbb{Z}^n} | \hat{b}(\xi)||\xi|^{-n} d\xi \lesssim 1.$$

Then, for x in $\{x \in \mathbb{T}^n : |x| \leq 2\}$ we can get

$$\int_{|x|\leq 2} | a(x,D)b(x)| dx \lesssim 1.$$

Now, we estimate the part where x in $S = \{x \in \mathbb{T}^n : |x| > 2\}$. As above when $r \geq 1$, we can write

$$\begin{aligned} |a(x,D)b(x)| &\lesssim |x|^{-2n} \sum_{l=0}^{2n} |\sum_{\xi\in\mathbb{Z}^n} e^{ix\xi} \Delta_\xi^{2n-l} a(x,\xi)\widehat{y^l b}(\xi)| \\ &= |x|^{-2n} \sum_{l=0}^{2n} | \tilde{a}_l(x,D)(y^l b)(x)|, \end{aligned}$$

where $\tilde{a}_l(x,D)$ is a pseudo-differential operator with the symbol $\Delta_\xi^{2n-l} a(x,\xi)$ belongs to $L^\infty S_0^{-n}$. From Lemma 1 we have that $\tilde{a}_l(x,D)$ is bounded from $L^{n/(n-1)}(\mathbb{T}^n)$ to $L^{n/(n-1)}(\mathbb{T}^n)$ if $n \geq 2$ and from $L^2(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$ if $n = 1$. Then, we have

$$\begin{aligned} \int_S | a(x,D)b(x)| dx &\lesssim \sum_{l=0}^{2n} \int_S | x|^{-2n} |\tilde{a}_l(y^l b)(x)| dx \\ &\lesssim 1 + \sum_{l=1}^{2n} \| y^l b \|_{L^{n/(n-1)}(\mathbb{T}^n)} \\ &\lesssim 1 + \sum_{l=1}^{2n} r^{l+\frac{n}{2}-1} \| b \|_{L^2(\mathbb{T}^n)} \lesssim 1. \end{aligned}$$

We use same computation for $n = 1$ by using L^2 boundedness.
The proof is completed.

Acknowledgments

Benhamoud Ramla was supported by the National Natural Science Foundation of China (No. 12071437).

Declarations

Conflict of interest The authors declare no competing interests.

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